

## Introduction

Polar coordinates provide an alternative way to describe the position of points in the plane. Instead of using horizontal and vertical distances (as in Cartesian coordinates), polar coordinates use distance from a fixed point and an angle.

**Fact** — A point in polar coordinates is described by  $(r, \theta)$  where:

- $r$  is the distance from the origin (pole) to the point
- $\theta$  is the angle measured anticlockwise from the positive  $x$ -axis (initial line)

### Example

Plot the points with polar coordinates  $(3, \frac{\pi}{4})$ ,  $(2, \pi)$ , and  $(4, \frac{3\pi}{2})$ .



## Converting Between Coordinate Systems

**Fact (Polar to Cartesian)** — If a point has polar coordinates  $(r, \theta)$ , then its Cartesian coordinates are:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

**Fact (Cartesian to Polar)** — If a point has Cartesian coordinates  $(x, y)$ , then its polar coordinates are:

$$r = \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x} \quad (\text{being careful about quadrants})$$

### Example

Convert the point with Cartesian coordinates  $(3, -4)$  to polar coordinates.

### Tip

There are multiple different polar coordinates which refer to the same point. To keep things clearer, we typically restrict the range of  $0 \leq \theta < 2\pi$ . Also the point  $(0, \theta)$

### Example

Convert the point with polar coordinates  $(6, \frac{2\pi}{3})$  to Cartesian coordinates.

## Equations in Polar and Cartesian Forms

**Example**

Find the polar equations of:

- $(x^2 + y^2)^2 = 4xy$
- $y = x^2$
- $x \cos \alpha + y \sin \alpha = p$

**Example**

Find the cartesian equations of the curves with the polar coordinates

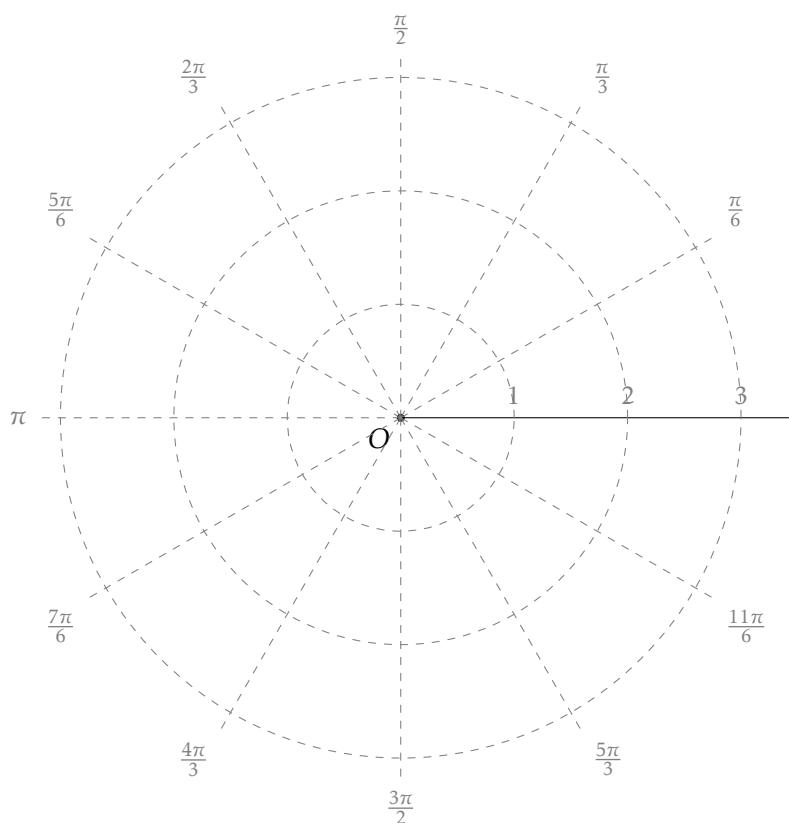
- $r = 2a \cos \theta$  for  $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$
- $r^2 = a^2 \sin 2\theta$

## Sketching Polar Curves

**Fact** — A polar equation has the form  $r = f(\theta)$  and describes a curve in the plane.

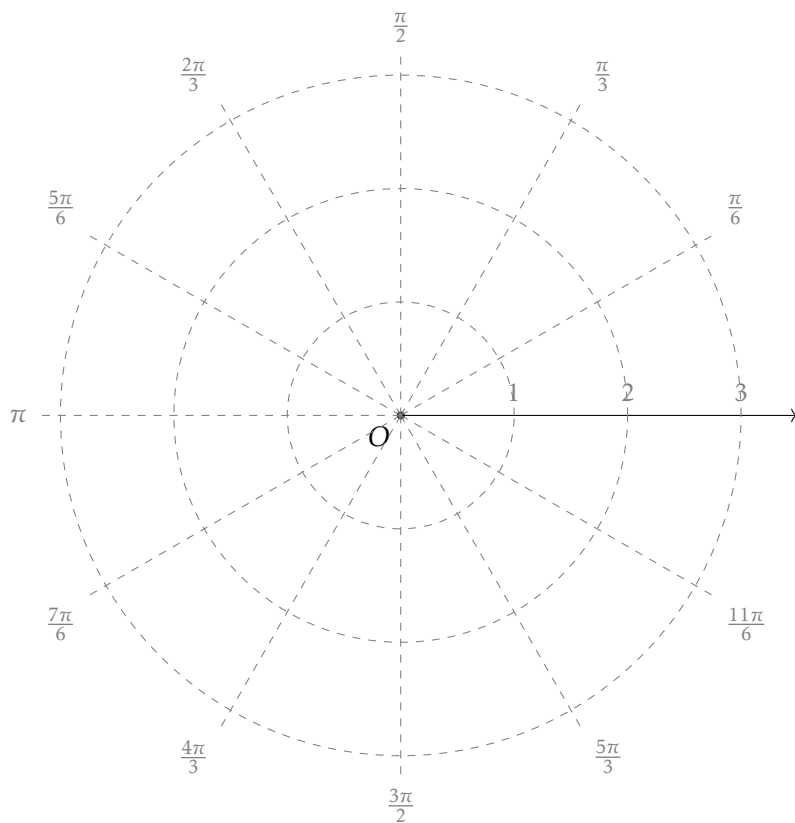
### Example

Sketch the graph of  $r = 2 + \cos \theta$  for  $0 \leq \theta \leq 2\pi$ .



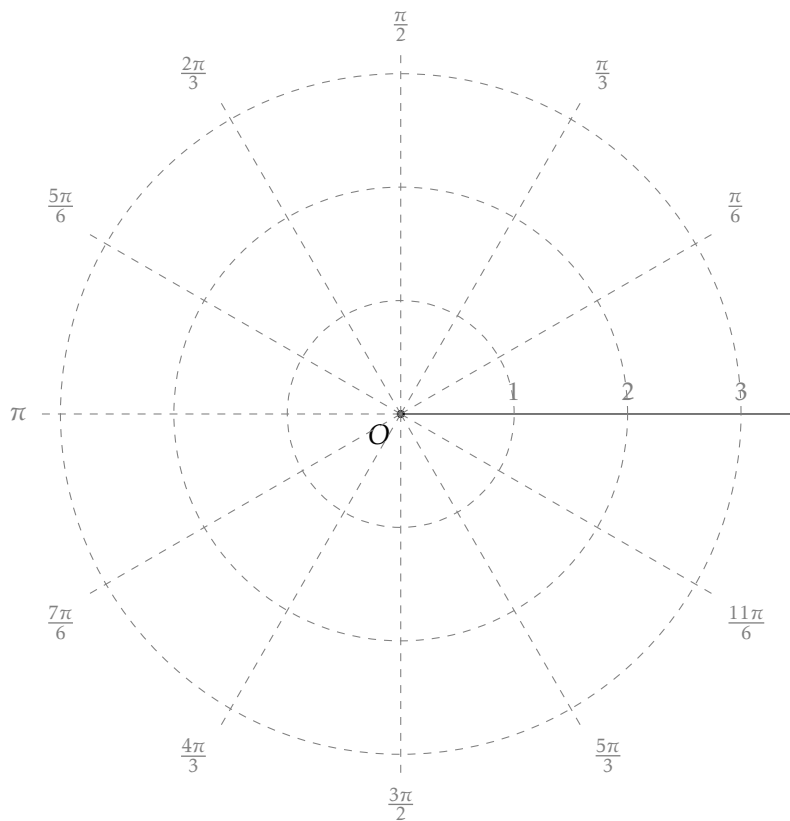
**Example**

Sketch the graph of  $r = 2\sin\theta$  for  $0 \leq \theta \leq \pi$ .



**Example**

Sketch the graph of  $r = 1 + \frac{\theta}{4}$  for  $0 \leq \theta \leq 2\pi$ .



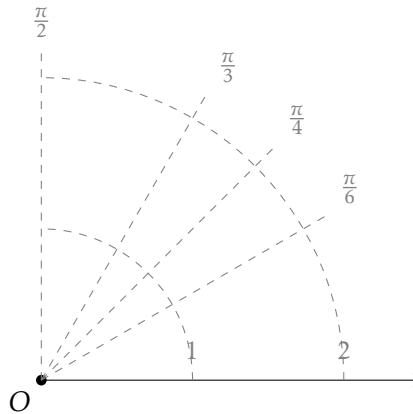
## Symmetry in Polar Graphs

**Fact (Tests for Symmetry)** — For a polar equation  $r = f(\theta)$ :

- **Symmetry about the x-axis:** If  $f(-\theta) \equiv f(\theta)$
- **Symmetry about the y-axis:** If  $f(\pi - \theta) \equiv f(\theta)$
- **Symmetry about the origin:** If  $f(\theta + \pi) \equiv f(\theta)$
- **Symmetry about the line  $\theta = \alpha$ :** If  $f(2\alpha - \theta) \equiv f(\theta)$

### Example

Draw the graph with equation  $r = 2 \sin 2\theta$  for  $0 \leq \theta \leq \frac{1}{2}\pi$ . Prove that it is symmetrical about the line  $\theta = \frac{\pi}{4}$



### Tip

When sketching polar curves:

1. Check for symmetry first
2. Find where  $r = 0$  (curves pass through origin)
3. Find maximum and minimum values of  $r$
4. Plot key points and use symmetry to complete the curve

## Greatest and least values of polar curves

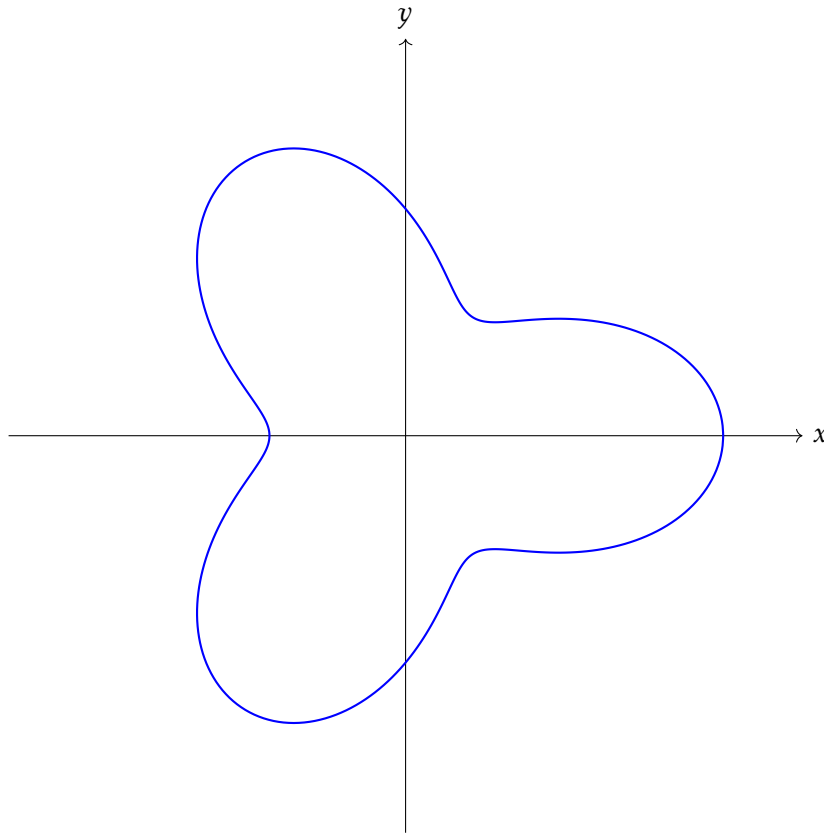
**Example**

For the graph with polar equation  $r = 3 + 2 \cos 3\theta$  with  $-\pi < \theta \leq \pi$ , find the greatest and least values of  $r$ , and the values of  $\theta$  for which they occur

**Example**

Find the maximum and minimum values of  $r$  for the curve with polar equation  $r = 2 + \cos \theta + \cos 2\theta$  for  $-\pi < \theta \leq \pi$ .

## Areas of Polar Curves



**Fact (Area Formula)** — The area enclosed by a polar curve  $r = f(\theta)$  from  $\theta = \alpha$  to  $\theta = \beta$  is:

**Example**

Find the area enclosed by the circle  $r = 4$ .

**Example**

Find the area enclosed by one petal of the rose curve  $r = 3 \sin(2\theta)$ .